1. Use induction to show that for all $n \geq 1$,

$$\lim_{n\to\infty} \frac{x^n}{e^x} = 0.$$

2. Evaluate the following limits.

   (a) $\lim_{n\to\infty} \frac{\log x}{x^p} = 0, \quad p > 0$

   (b) $\lim_{n\to 0} \frac{5x - \tan(5x)}{x^3}$

   (c) $\lim_{n\to 1} \frac{1}{\log x} - \frac{1}{x-1}$

3. **(Hard)** Evaluate the following limits.

   (a) $\lim_{n\to 0^+} x^x$

   (b) $\lim_{n\to\infty} \left(1 + \frac{1}{x}\right)^x$

   (c) $\lim_{x\to 0} (1 + \sin(2x))^{\cot(3x)}$

   Note: These will not be tested.

Solution.

1. Let's check the base case, when $n = 1$. In this case, we have a limit of the form $\frac{\infty}{\infty}$, so by L'Hôpital's rule,

$$\lim_{x\to\infty} \frac{x}{e^x} = \lim_{x\to\infty} \frac{1}{e^x} = 0$$

Thus the result is true for $n = 1$. Now let's suppose there is a $k \geq 1$ such that

$$\lim_{x\to\infty} \frac{x^k}{e^x} = 0.$$

Now since $\frac{x^{k+1}}{e^x}$ is of the form $\frac{\infty}{\infty}$ as x goes to infinity, by L'Hôpital's rule,

$$\lim_{x\to\infty} \frac{x^{k+1}}{e^x} = \lim_{x\to\infty} \frac{(k+1)x^k}{e^x} = (k+1) \lim_{x\to\infty} \frac{x^k}{e^x} = 0.$$

Thus by induction, we have for all $n \geq 1$,

$$\lim_{x\to\infty} \frac{x^n}{e^x} = 0.$$

2. (a) $\frac{\log x}{x^p}$ is of the form $\frac{\infty}{\infty}$ as x goes to infinity, so by L'Hôpital's rule,

$$\lim_{x\to\infty} \frac{\log x}{x^p} = \lim_{x\to\infty} \frac{\frac{1}{x}}{px^{p-1}} = \lim_{x\to\infty} \frac{1}{x^p} = 0$$

The last line is true since $p > 0$.

(b) Our limit is of the form $\frac{0}{0}$ as x goes to 0.

$$\lim_{x\to 0} \frac{5x - \tan(5x)}{x^3} = \lim_{x\to 0} \frac{5 - 5\sec^2(5x)}{3x^2}$$

Our new limit is again of the form $\frac{0}{0}$, so let's use L'Hôpital's Rule again,

$$\begin{aligned}
\lim_{x\to 0} \frac{5 - 5\sec^2(5x)}{3x^2} &= \lim_{x\to 0} \frac{-5 \cdot 2\sec(5x)\sec(5x)\tan(5x)5}{6x} \\
&= \lim_{x\to 0} \frac{-25\sec^2(5x)}{3\cos(5x)} \lim_{x\to 0} \frac{\sin(5x)}{x} \\
&= -\frac{25}{3} \lim_{x\to 0} 5\frac{\sin(5x)}{5x} \\
&= -\frac{125}{3}
\end{aligned}$$

Where in the last line we used the fact that

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

(c) We begin by simplifying.

$$\frac{1}{\log x} - \frac{1}{x-1} = \frac{x-1-\log x}{(x-1)\log x}$$

Which is of the form $\frac{0}{0}$ as x goes to 1, so let's use L'Hôpital.

$$\begin{aligned}\lim_{x\to 1} \frac{x-1-\log x}{(x-1)\log x} &= \lim_{x\to 1} \frac{1-\frac{1}{x}}{\log x + \frac{x-1}{x}} \\ &= \lim_{x\to 1} \frac{x-1}{x\log x + x - 1}\end{aligned}$$

Which is of the form $\frac{0}{0}$, so let's use L'Hôpital's rule again.

$$\begin{aligned}\lim_{x\to 1} \frac{x-1}{x\log x + x - 1} &= \lim_{x\to 1} \frac{1}{\log x + x\frac{1}{x} + 1} \\ &= \lim_{x\to 1} \frac{1}{\log x + 2} \\ &= \frac{1}{2}\end{aligned}$$

3. (a) Again let $L = \lim_{x\to 0^+} x^x$, which is of the form 0^0 so we need to do some work. Let's begin by taking logarithms of both sides,

$$\begin{aligned}\log L &= \log\left(\lim_{x\to 0^+} x^x\right) \\ &= \lim_{x\to 0^+} \log x^x \\ &= \lim_{x\to 0^+} x\log x \\ &= \lim_{x\to 0^+} \frac{\log x}{\frac{1}{x}}\end{aligned}$$

The second line is true by the continuity of log. So now we have a limit of the form $\frac{-\infty}{\infty}$, so we can apply L'Hôpital's rule.

$$\begin{aligned}\lim_{x\to 0^+} \frac{\log x}{\frac{1}{x}} &= \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} \\ &= \lim_{x\to 0^+} -x \\ &= 0\end{aligned}$$

So we have $\log L = 0$ and thus $L = 1$, i.e.,

$$\lim_{x\to 0^+} x^x = 1.$$

(b) Let $L = \lim_{x\to\infty}(1+1/x)^x$, which is of the form 1^∞. Again let's take log of both sides, and the continuity of logarithm gives us,

$$\begin{aligned}\log L &= \lim_{x\to\infty} x \log\left(1+\frac{1}{x}\right) \\ &= \lim_{x\to\infty} \frac{\log\left(1+\frac{1}{x}\right)}{\frac{1}{x}}\end{aligned}$$

Now we have a limit of the form $\frac{0}{0}$, so by L'Hôpitals rule, we have

$$\begin{aligned}\lim_{x\to\infty} \frac{\log\left(1+\frac{1}{x}\right)}{\frac{1}{x}} &= \lim_{x\to\infty} \frac{\frac{1}{1+\frac{1}{x}}\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}} \\ &= \lim_{x\to\infty} \frac{1}{1+\frac{1}{x}} \\ &= 1\end{aligned}$$

So $\log L = 1$, and so $L = e$, i.e.,

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e.$$

(c) Again let $L = \lim_{x\to 0}(1+\sin(2x))^{\cot(5x)}$, which is of the form 1^∞. Again as in (a), (b), let's take logarithms.

$$\begin{aligned}\log L &= \lim_{x\to 0} \cot(5x)\log(1+\sin(2x)) \\ &= \lim_{x\to 0} \frac{\log(1+\sin(2x))}{\tan(5x)}\end{aligned}$$

Which is now of the form $\frac{0}{0}$. So by L'Hôpital's rule,

$$\begin{aligned}\lim_{x\to 0} \frac{\log(1+\sin(2x))}{\tan(5x)} &= \lim_{x\to 0} \frac{\frac{2\cos(2x)}{1+\sin(2x)}}{5\sec^2(5x)} \\ &= \frac{\frac{2\cos 0}{1+\sin 0}}{5\sec^2 0} \\ &= \frac{2}{5}\end{aligned}$$

Thus $\log L = \frac{2}{5}$ and therefore $L = e^{\frac{2}{5}}$, i.e.,

$$\lim_{x\to 0}(1+\sin(2x))^{\cot(5x)} = e^{\frac{2}{5}} \quad \text{(wtf...)}$$